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SOLUTION OF CERTAIN BOUNDARY PROBLEMS OF MATHEMATICAL PHYSICS BY THE COLLOCATION METHOD

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ABSTRACT: Presented in this article is a very simple and effective means of solving a number of boundary problems of mathematical physics by the collocation method. Questions concerning existence, and convergence of approximate solutions found with this method are discussed in the work. Estimates of the speed of convergence of approximate solutions on the exact solution are included.

INTRODUCTION

The collocation method, or interpolation method, is mathematically simple /3* and requires no special preliminary information; at the same time it is an effective means of solving various problems in mathematical physics. This method is promising from the standpoint of computer technology, since it requires very little manual labor. Meanwhile much less attention has been devoted to it in the mathematical literature than to other methods.

The solution v of the differential equation describing some physical process whould be determined according to given functions f. Information about functions f derived from experiment is usually presented in tabular form. This is very convenient in terms of application of the collocation method. Furthermore the approximate solution found by the interpolation method is polynomial in terms of the corresponding variables, which is useful in theoretical analysis.

Questions of the existence, convergence and speed of conversions of approximate solutions in the case of boundary problems for the elliptical and parabolic

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*Numbers in the amrgin indicate pagination in the foreign text.

equations, stationary and non-stationary Navier-Stokes equation system of a viscous incompressible fluid are discussed in this article. /4

1. Interpolation method of solving boundary problems for elliptical equations.

We will examine the first boundary for the equation

$$\Delta_{\rho,\varphi} u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = f\left(\rho,\varphi,\frac{\partial u}{\partial\rho},\frac{\partial u}{\partial\varphi},u\right) \tag{1}$$

in a circle Ω of radius R. Here ρ,ϕ are polar coordinates. At the boundary of the circle Ω

$$u|_{\rho=R} = 0 \tag{2}$$

is satisfied.

The approximate solution of problem (1,2) is sought in the form

$$u_n(\rho,\varphi) = \frac{1}{2n+1}\sum_{m=0}^{2n} a_m^{(n)}(\rho)\,\frac{\sin\,(2n+1)\frac{\varphi-\varphi_m}{2}}{\sin\frac{\varphi-\varphi_m}{2}}$$

The collocation method consists in the fact that the unknown functions

$$a_m^{(n)}(\rho),\ m=0,\dots,2n.$$

are determined from a system of 2n + 1 equations.

$$\left[\Delta_{\rho,\varphi} u_n - f\left(\rho,\varphi,\frac{\partial u_n}{\partial\rho},\frac{\partial u_n}{\partial\varphi},u_n\right)\right]_{\substack{\varphi=\varphi_m\\ 0\le\rho\le R}} = 0, \tag{3}$$

$$u|_{\rho=R} = 0, \tag{4}$$

where $\varphi_m,\ m=0,1,\dots,2n,$ –

are fixed numbers, called nodes of interpolation. We will assume

$$\varphi_m = \frac{2m\pi}{2n+1},\quad m=0,1,\dots,2n\ .$$

We shall study the problems of convergence and rate of convergence of the approximate solutions obtained by this method to the exact solution. /5

We shall assume that there exists a solution $u^*(\rho,\varphi)$ problem (1,2) twice continuously differentiable in terms of (ρ,ϕ) in $\bar{\Omega}$.

The following assumptions are made: 1) the function $f(\rho,\varphi,\frac{\partial u}{\partial \rho},\frac{\partial u}{\partial \varphi},u)$ is Holder--continuous with index $\delta, 0<\delta\leq 1,$ relative to (ρ,ϕ) uniformly in terms of $(\frac{\partial u}{\partial \rho},\frac{\partial u}{\partial \varphi},u)$ the range $V=\{(\rho,\varphi,\frac{\partial u}{\partial \rho},\frac{\partial u}{\partial \varphi},u):$ now where $(\rho,\varphi)\in\bar{\Omega}, |u-u^*(\rho,\varphi)|\leq q_1, |\frac{\partial u}{\partial \rho}-\frac{\partial u^*}{\partial \rho}(\rho,\varphi)|\leq q_2, |\frac{\partial u}{\partial \varphi}-\frac{\partial u^*}{\partial \varphi}(\rho,\varphi)|\leq q_3\},$ are q_1, q_2, q_3 constants; 2) The functions $f'_{\frac{\partial u}{\partial \rho}}(\rho,\varphi,\frac{\partial u}{\partial \rho},\frac{\partial u}{\partial \varphi},u), f'_{\frac{\partial u}{\partial \varphi}}(\rho,\varphi,\frac{\partial u}{\partial \rho},\frac{\partial u}{\partial \varphi},u), f'_u(\rho,\varphi,\frac{\partial u}{\partial \rho},\frac{\partial u}{\partial \varphi},u)$ are defined and continuous in range V; 3) The homogeneous problem

$$\Delta_{\rho,\varphi} u = f'_{\frac{\partial u}{\partial \rho}}(\rho,\varphi,\frac{\partial u^*}{\partial \rho}(\rho,\varphi),\frac{\partial u^*}{\partial \varphi}(\rho,\varphi),u^*(\rho,\varphi))\frac{\partial u}{\partial \rho} + f'_{\frac{\partial u}{\partial \varphi}}(\rho,\varphi,\frac{\partial u^*}{\partial \rho}(\rho,\varphi),\frac{\partial u^*}{\partial \varphi}(\rho,\varphi),u^*(\rho,\varphi))\frac{\partial u}{\partial \varphi} + f'_u(\rho,\varphi,\frac{\partial u^*}{\partial \rho}(\rho,\varphi),\frac{\partial u^*}{\partial \varphi}(\rho,\varphi),u^*(\rho,\varphi))u, \quad (\rho,\varphi)\in\Omega,$$

$$u|_{\rho=R}=0$$

has only a zero solution.

Analysis of convergence is based on the results of the theory of projection methods [1]. We denote the following:

$$z^*(\rho,\varphi)\equiv\Delta_{\rho,\varphi}u^*(\rho,\varphi), \qquad (\rho,\varphi)\in\bar{\Omega},$$
$$z_n(\rho,\varphi)\equiv\Delta_{\rho,\varphi}u_n(\rho,\varphi), \qquad (\rho,\varphi)\in\bar{\Omega}.$$

In view of the assumptions relative to solution $u^*(\rho,\phi)$ the function $z^*(\rho,\varphi)$ is continuous in $\bar{\Omega}$.

We will introduce the following Banach spaces. $C(\bar{\Omega})$ is the space of the functions $u(\rho,\phi)$, non-continuous in $\bar{\Omega}$ with the standard $\|u\|_{C(\bar{\Omega})}=\max_{(\rho,\varphi)\in\bar{\Omega}}|u(\rho,\varphi)|$. $u(\rho,\phi)$, is the space of functions $\mathring{H}_{1+\delta}(\bar{\Omega})$ - , equal to zero when $\rho = R$, continuous along with $\frac{\partial u}{\partial \rho}(\rho,\varphi), \frac{\partial u}{\partial \varphi}(\rho,\varphi)$ with bounded standard $\|u\|_{\mathring{H}_{1+\delta}(\bar{\Omega})}$,

$$\|u\|_{\mathring{H}_{1+\delta}(\bar{\Omega})}=|u|_\delta+|\frac{\partial u}{\partial \rho}|_\delta+|\frac{\partial u}{\partial \varphi}|_\delta,$$

where

$$|v|_\delta=|v|_0+\langle v\rangle_\delta, \quad |v|_0=\max_{(\rho,\varphi)\in\bar{\Omega}}|v(\rho,\varphi)|,$$

$$\langle v\rangle_\delta=\sup_{(\rho,\varphi),(\rho^\circ,\varphi^\circ)\in\bar{\Omega}}\frac{|v(\rho,\varphi)-v(\rho^\circ,\varphi^\circ)|}{(\sqrt{(\rho-\rho^\circ)^2+(\varphi-\varphi^\circ)^2})^\delta},$$

$$0<\delta\leq 1.$$

$H_\delta(\bar{\Omega})$ - is the space of continuous functions with bounded standard $\|u\|_{H_\delta(\bar{\Omega})} = |u|_0 + \langle u \rangle_\delta, \quad 0 < \delta \leq 1.$

We will assume that for $n, n = 1, 2, \ldots,$ unique approximate solutions exist $u_n(\rho, \phi)$ with bounded standard $\|u\|_{\tilde{H}_{1+\delta}(\bar{\Omega})}$.

Let us examine the following boundary problem:

$$\Delta_{\rho,\varphi} u = z(\rho, \varphi), \qquad (\rho, \varphi) \in \Omega, \qquad u|_{\rho=R} = 0, \tag{6}$$

where the function $C(\bar{\Omega})$ is a set of space $z(\rho, \varphi)$.

Let us switch to Cartesian coordinates:

$$\Delta_{x_1, x_2} v(x_1, x_2) = \frac{\partial^2}{\partial x_1^2} v(x_1, x_2) + \frac{\partial^2}{\partial x_2^2} v(x_1, x_2) = \tilde{z}(x_1, x_2), \quad (x_1, x_2) \in \Omega, \qquad v|_{x_1^2 + x_2^2 = R^2} = 0, \tag{6^1}$$

where $\tilde{z}(x_1, x_2) = [z(\rho, \varphi)]_{\rho = \sqrt{x_1^2 + x_2^2},\ \varphi = \arctan \frac{x_2}{x_1}},$

$v(x_1, x_2) = [u(\rho, \varphi)]_{\rho = \sqrt{x_1^2 + x_2^2},\ \varphi = \arctan \frac{x_2}{x_1}}.$

It is clear that $\tilde{z}(x_1, x_2)$ is a non-continuous function in $\bar{\Omega}$.

Boundary problem (6^1), if $\tilde{z}(x_1, x_2) \in L_p(\Omega)$, has a unique generalized solution $v(x_1, x_2) \in W_p^{(2)}(\Omega)$, and the estimate $\|v\|_{W_p^{(2)}(\Omega)} \leq q_4 \|\tilde{z}\|_{L_p(\Omega)}$, is valid, where q_4 is a constant, $p > 1$ [2].

Definitions of spaces $L_p(\Omega), W_p^{(2)}(\Omega)$ can be found, for instance, in [3]. /7

In view of the enclosure theorem [3] $\|v\|_{\tilde{H}_{1+\delta}(\bar{\Omega})} \leq q_5 \|v\|_{W_p^{(2)}(\Omega)}$, the enclosure operator is completely non-continuous when $\delta < \frac{2p-2}{p}$, q_5, is a constant.

$\tilde{H}_{1+\delta}(\bar{\Omega})$ - is a space of functions $v(x_1, x_2)$, equal to zero when $x_1^2 + x_2^2 = R^2$, continuous along with $\frac{\partial v}{\partial x_1}(x_1, x_2)$, $\frac{\partial v}{\partial x_2}(x_1, x_2)$ in $\bar{\Omega}$ with limited bound $\|v\|_{\tilde{H}_{1+\delta}(\bar{\Omega})}$. Bound $\|v\|_{\tilde{H}_{1+\delta}(\bar{\Omega})}$, like $\|u\|_{\tilde{H}_{1+\delta}(\bar{\Omega})}$, is determined with substitutions of the symbols ρ, ϕ for x_1, x_2.

From the last estimate we have: $\|u\|_{\tilde{H}_{1+\delta}(\bar{\Omega})} \leq q_6 \|v\|_{\tilde{H}_{1+\delta}(\bar{\Omega})} \leq$

$$\leq q_5 q_6 \|v\|_{W_p^{(2)}(\Omega)} \leq q_4 q_5 q_6 \|\tilde{z}\|_{L_p(\Omega)} \leq q_4 q_5 q_6 q_7 \|z\|_{C(\bar{\Omega})}, \tag{7}$$

where q_6, q_7 are constants.

In other words, there exists a linear, completely continuous operator A, acting from $C(\bar{\Omega})$ on $H_{1+\delta}(\bar{\Omega})$ with bound $\|A\| \leqslant q_4 q_5 q_6 q_7$.

Thus, if the functions $z_n(\rho,\varphi)$, defined by relationships 5, n = 1,2,..., belong to $C(\bar{\Omega})$, then for functions $u^*(\rho,\varphi)-u_n(\rho,\varphi)$, $z^*(\rho,\varphi)-z_n(\rho,\varphi)$ in view of (7), the inequalities

$$\|u_n - u^*\|_{H_{1+\delta}(\bar{\Omega})} \leqslant q_4 q_5 q_6 q_7 \|z_n - z^*\|_{C(\bar{\Omega})}. \tag{8}$$

will be satisfied.

Since operator A is bounded we may take in space $C(\bar{\Omega})$ a sphere $\|z - z^*\|_{C(\bar{\Omega})} \leqslant \sigma_0$ with radius σ_0 so small that the functions $u(\rho,\phi) = AZ(\rho,\phi)$, $\|z - z^*\|_{C(\bar{\Omega})} \leqslant \sigma_0$, will satisfy the inequalities

$$|u - u^*(\rho,\varphi)| \leqslant q_0,\ \left|\frac{\partial u}{\partial \rho} - \frac{\partial u^*}{\partial \rho}(\rho,\varphi)\right| \leqslant q_1,\ \left|\frac{\partial u}{\partial \varphi} - \frac{\partial u^*}{\partial \varphi}(\rho,\varphi)\right| \leqslant q_2,\quad (\rho,\varphi) \in \bar{\Omega}.$$

We will proceed from (1), (2) to the task of finding the function $z^*(\rho,\varphi)$ belonging to space $C(\bar{\Omega})$, satisfying the equation

$$z(\rho,\varphi) = f\left(\rho,\varphi,\frac{\partial}{\partial\rho}Az(\rho,\varphi),\frac{\partial}{\partial\varphi}Az(\rho,\varphi),Az(\rho,\varphi)\right) = PBz(\rho,\varphi). \tag{9}$$

Here P is the linear bounded operator of enclosure of $H_\delta(\bar{\Omega})$ and $C(\bar{\Omega})$, /8

$$B = f\left(\rho,\varphi,\frac{\partial}{\partial\rho}A\ldots,\frac{\partial}{\partial\varphi}A\ldots,A\ldots\right)$$

is the operator acting from set $V\left\{Z;\ \|Z - Z^*\|_{C(\bar{\Omega})} \leqslant \leqslant \sigma_0\right\} \subset C(\bar{\Omega})$ in space $H_\delta(\bar{\Omega})$. Operator B is completely continuous in set V in view of the perfect continuity of operator A and condition 1).

From (3), (4) of determning the approximate solution $u_n(\rho,\phi)$ we proceed to the problem of finding the function that satisfies the operator equation

$$P_n(z_n - Bz_n) = 0, \tag{10}$$

where P_n is the projector that places in correspondence each function $\Psi(\rho,\phi)$ continuous with respect to ϕ according to its trigonometric interpolation polynomial of the order n with nodes $\phi_m^{(n)}$, m = 0.1...., 2n, in terms of the variable ϕ. However $z_n(\rho,\varphi) = \Delta_{\rho,\varphi} u_n(\rho,\varphi)$ - is a trigonometric polynomial of order $\leqslant n$ in terms of ϕ. This means $P_n Z_n = Z_n$, and we proceed from (10) to the equation

$$z_n = P_n B z_n . \qquad (11)$$

Note that P_n is a linear bounded operator acting from $H_\delta(\bar{\Omega})$ on $C(\bar{\Omega})$.

According to the interpolation theorem [4], for any $Z \in H_\delta(\bar{\Omega})$ we shall have $\|P_n z - Pz\|_{C(\bar{\Omega})} \to 0$ with $n \to +\infty$.

In view of condition (2) the operator PB is continuously Freshe-differentiable at the point $Z^*(\rho,\phi)$ in space $C(\bar{\Omega})$. We will prove that the homogeneous equation $h = PB'(Z^*)h$ has only a trival solution. This equation is equivalent to finding the solution $u(\rho,\phi)$ of the problem

$$\Delta_{\rho,\varphi} u = -f'_{\frac{\partial u}{\partial \rho}}\left(\rho,\varphi,\frac{\partial u^*}{\partial \rho}(\rho,\varphi),\frac{\partial u^*}{\partial \varphi}(\rho,\varphi),u^*(\rho,\varphi)\right)\frac{\partial u}{\partial \rho} + f'_{\frac{\partial u}{\partial \varphi}}\left(\rho,\varphi,\frac{\partial u^*}{\partial \rho}(\rho,\varphi),\frac{\partial u^*}{\partial \varphi}(\rho,\varphi),u^*(\rho,\varphi)\right)\frac{\partial u}{\partial \varphi} + f'_u\left(\rho,\varphi,\frac{\partial u^*}{\partial \rho}(\rho,\varphi),\frac{\partial u^*}{\partial \varphi}(\rho,\varphi),u^*(\rho,\varphi)\right)u,$$
$$u|_{\rho=R} = 0 .$$ /9

This problem, according to the proposition (3), has only a zero solution.

All conditions of the theorem of convergence of the approximate solutions on the exact [1, pp. 293-294] are satisfied. We shall present this theorem here.

Theorem. Let operator B be completely continuous on set U of Banach space $C(\bar{\Omega})$, and let equation $Z = PB_Z$ have the isolated solution $Z^* \in V$ with a zero component. Let the projectors P_n be bounded as operators from Banach space $H_\delta(\bar{\Omega})$ to Banach space $C(\bar{\Omega})$ and $P_n \to P$ strongly with $n \to +\infty$.

Then we find those n_0, σ_1 for which with $n \geqslant n_0$ the equation $Z_n = P_n B Z_n$ has in sphere $\|z - z^*\|_{C(\bar{\Omega})} \leqslant \sigma_1$ only one solution z_n, and all such solutions z_n for $n \to +\infty$ according to the bounds of space $C(\bar{\Omega})$ approach $Z^*(\rho,\phi)$. If operator B is Freshe-differentiable at point z^* and the homogeneous equation $h = PB'(Z^*)h$ has only a zero solution[1], then the estimate of convergence is valid: $q_8 \|Pz^* - P_n z^*\|_{C(\bar{\Omega})} \leqslant \|z_n - z^*\|_{C(\bar{\Omega})} \leqslant q_9 \|Pz^* - P_n z^*\|_{C(\bar{\Omega})}$, where q_8, q_9 are certain constants.

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[1]Hence follows the isolation of z* and the non zero value of the exponent.

If, moreover, operator B is continuously Freshe-differentiable at point z^*, then for sufficiently large n the solution z_n of equation $Z_n = P_n BZ_n$ is unique in the sphere $\|z - z^*\|_{C(\bar{\Omega})} \leq \sigma_2$ of sufficiently small radius σ_2, $\sigma_z \leq \sigma_1$.

In view of interpolation theorem [4], $\|z^* - P_n z^*\|_{C(\bar{\Omega})} \leq E_n(z^*(\rho,\varphi))(q_{10} + q_{11}\ln n)$, /10
where $E_n(z^*(\rho,\varphi)) = \sup_{0\leq\rho\leq R} E_n^{\varphi}(z^*(\rho,\varphi))$, $E_n^{\varphi}(z^*(\rho,\varphi))$ – is the best uniform approximation of the function $Z^*(\rho,\phi)$ by trigonometric polynomials of order not exceeding n in terms of variable ϕ for fixed ρ, $0 \leq \rho \leq R$; q_{10}, q_{11} are absolute constants.

For any n solution $Z_n(\rho,\phi)$ of problem (11) corresponds to solution $U_n(\rho,\phi)$ of problem (3), (4) with bound $\|u_n\|_{H^2_{1+\delta}(\bar{\Omega})}$, where $\|u_n - u^*\|_{H^2_{1+\delta}(\bar{\Omega})} \leq q_4 q_5 q_6 q_7 \|z_n - z^*\|_{C(\bar{\Omega})}$.

Thus, the following is valid:

Theorem 1. Let conditions 1), 2), 3), be satisfied. Then we also find number n_0, σ_2, such that for $n \geq n_0$ the solution $U_n(\rho,\phi)$ of problem (3), and (4) belongs to sphere $\|u - u^*\|_{H^2_{1+\delta}(\bar{\Omega})} \leq q_4 q_5 q_6 q_7 \sigma_2$ and the estimates

$$\|u_n - u^*\|_{H^2_{1+\delta}(\bar{\Omega})} \leq q_4 q_5 q_6 q_7 q_9 E_n(\Delta_{\rho,\varphi} u^*(\rho,\varphi))(q_{10} + q_{11}\ln n),$$
$$q_8 \|\Delta_{\rho,\varphi} u^* - P_n \Delta_{\rho,\varphi} u^*\|_{C(\bar{\Omega})} \leq \|\Delta_{\rho,\varphi} u^* - \Delta_{\rho,\varphi} u_n\|_{C(\bar{\Omega})} \leq$$
$$\leq q_9 E_n(\Delta_{\rho,\varphi} u^*(\rho,\varphi))(q_{10} + q_{11}\ln n).$$

are satisfied.

Note 1. If the function $\Delta_{\rho,\phi} u^*(\rho,\phi)$ has a continuous derivative $\frac{\partial^s}{\partial \varphi^s}[\Delta_{\rho,\varphi} u^*(\rho,\varphi)]$, $s = 0, 1, 2, \ldots$, satisfying in terms of the argument ϕ the Holder condition with α, $0 < \alpha \leq 1$ uniformly in terms of ρ, then according to the Jackson theorem [4] $E_n(\Delta_{\rho,\varphi} u^*(\rho,\varphi)) \leq q_{12} q_{13} \frac{(s+1)^{s+1}}{(s+1)!}\left(\frac{1}{n}\right)^s \left(\frac{2}{n-s}\right)^{\alpha}$ where q_{12} is the Holder constant of the function, $\frac{\partial^s}{\partial \varphi^s}[\Delta_{\rho,\varphi} u^*(\rho,\varphi)]$, q_{13} is an absolute constant. Consequently we have the estimates

$$\|u_n - u^*\|_{H^2_{1+\delta}(\bar{\Omega})} \leq q_4 q_5 q_6 q_7 q_9 q_{12} q_{13} \frac{(s+1)^{s+1}}{(s+1)!}\left(\frac{2}{n-s}\right)^{\alpha} \frac{(q_{10} + q_{11}\ln n)}{n^s},$$

$$\|\Delta_{\rho,\varphi} u_n - \Delta_{\rho,\varphi} u^*\|_{C(\bar{\Omega})} \leq q_9 q_{12} q_{13} \frac{(s+1)^{s+1}}{(s+1)!}\left(\frac{2}{n-s}\right)^{\alpha} \frac{(q_{10} + q_{11}\ln n)}{n^s}.$$ /11

Note 2. We may apply the method of collocation to the solution of the problem

$$\begin{cases} \Delta_{\rho,\varphi} u = f\left(\rho, \varphi, \dfrac{\partial u}{\partial \rho}, \dfrac{\partial u}{\partial \varphi}, u\right); & (\rho,\varphi) \in \Omega, \\ \left[\dfrac{\partial u}{\partial \rho} + \beta u\right]_{\rho = R} = 0, & \\ \beta > 0. & \end{cases}$$

If conditions similar to 1), 2), 3), are satisfied we also obtain the same estimates of conversions of the method as we did above.

Note 3. All the results apply without change to the case when

$$\bar{\Omega} = ((\rho,\varphi):\ 0 \leq \varphi < 2\pi,\ 0 < R_1 \leq \rho \leq R_2).$$

2. Application of the Collation Method to a stationary linearized boundary problem for Navier-Stokes equations of a Viscous incompressible fluid.

In this section we will use the interpolation method to find the approximate solution of the Stokes problem specifically: Hydrodynamic velocity $\vec{v}(\rho,\varphi)$ and pressure p are determined in bounded range Ω, now belonging to two dimensional Euclidean space from the conditions:

$$\nu \Delta_{\rho,\varphi} \vec{v} = \text{deg}_{\rho,\varphi}\, p + f(\rho,\varphi), \quad (\rho,\varphi) \in \Omega, \tag{12}$$

$$\text{div}_{\rho,\varphi} \vec{v} = 0, \quad (\rho,\varphi) \in \Omega, \tag{13}$$

$$\vec{v}|_S = (0,0). \tag{14}$$

Here S is the boundary of region Ω; ρ,ϕ are polar coordinates, ν is the Kinematic coefficient of viscosity. $\Delta_{\rho,\varphi} \vec{v} = \text{deg}_{\rho,\varphi}\, \text{div}_{\rho,\varphi} \vec{v} - \text{curl}_{\rho,\varphi}\, \text{curl}_{\rho,\varphi} \vec{v}$.

For the unique solution (1), (2), (3) we require that (15)

$$\int_S p\, ds = 0.$$

The approximate solution of problem (12) - (15) is sought in the form:

$$v_\rho^{(n)}(\rho,\varphi) = \frac{1}{2n+1} \sum_{m=0}^{2n} \alpha_m^{(n)}(\rho) \frac{\sin (2n+1)\frac{\varphi - \varphi_m^{(n)}}{2}}{\sin \frac{\varphi - \varphi_m^{(n)}}{2}},$$

$$v_\varphi^{(n)}(\rho,\varphi) = \frac{1}{2n+1} \sum_{m=0}^{2n} \beta_m^{(n)}(\rho) \frac{\sin (2n+1)\frac{\varphi - \varphi_m^{(n)}}{2}}{\sin \frac{\varphi - \varphi_m^{(n)}}{2}},$$

/12

$$p^{(n)}(\rho,\varphi) = \frac{1}{2n+1} \sum_{m=0}^{2n} c_m^{(n)}(\rho) \frac{\sin (2n+1)\frac{\varphi - \varphi_m^{(n)}}{2}}{\sin \frac{\varphi - \varphi_m^{(n)}}{2}}$$

The unknown functions $a_m^{(n)}(\rho), b_m^{(n)}(\rho), c_m^{(n)}(\rho), m=0,1,\ldots,2n,$ according to the collocation method, are determined from a system $2(2n+1)$ of ordinary differential equations $\left[\nu \Delta_{\rho,\varphi} \vec{z}^{(n)} - \operatorname{deg}_{\rho,\varphi} p^{(n)} - \vec{f}\,\right]_{\substack{\varphi=\varphi_m^{(n)} \\ \rho \in \bar{\Omega}}} = (0,0)$ (16)

and relations

$$\left[\operatorname{div}_{\rho,\varphi} \vec{z}^{(n)}\right]_{\substack{\varphi=\varphi_m^{(n)} \\ \rho \in \bar{\Omega}}} = 0, \tag{17}$$

$$\vec{z}^{(n)}\Big|_{\substack{\varphi=\varphi_m^{(n)} \\ \rho \in S}} = (0,0), \quad m=0,1,\ldots,2n, \tag{18}$$

$$\int_S p^{(n)}\, ds = 0. \tag{19}$$

The convergence of the collocation method depends partially on the choice of nodes of collocation $\varphi_m^{(n)}, \; m=0,1,\ldots,2n$.

We will assume $\varphi_m^{(n)} = \frac{2\pi m}{2n+1}, \; m=0,1,\ldots,2n$.

In analysing the convergence of the method we will assume that region Ω is a circle, i.e. $\bar{\Omega} = \{(\rho,\varphi):\; 0 \le R_1 \le \rho \le R_2,\; 0 \le \varphi < 2\pi\}$.

The functions $z_\rho^{(n)}(\rho,\varphi), z_\varphi^{(n)}(\rho,\varphi), \operatorname{div}_{\rho,\varphi} \vec{z}^{(n)}(\rho,\varphi)$ are trigonometric polynomials of an order not exceeding n in terms of variable $\varphi, 0 \le \varphi < 2\pi$, which have exactly 2n roots.

Because of this and because of the fact that Ω is a circle, relations (17), (18), (19) are valid for all $\phi \varepsilon (o, 2n)$. /13

We will denote by $C(\bar{\Omega})$ the space of continuous Vector-functions $\vec{z}(\rho,\varphi) = (z_\rho(\rho,\varphi), z_\varphi(\rho,\varphi))$ with bound $\|\vec{z}\|_{C(\bar{\Omega})} = \max_{(\rho,\varphi)\in\bar{\Omega}} |\vec{z}(\rho,\varphi)|$, where $|\vec{z}(\rho,\varphi)| = \max\{|z_\rho(\rho,\varphi)|, |z_\varphi(\rho,\varphi)|\}$. $\mathring{H}_{1+\delta}(\bar{\Omega})$ is a space of Vector-functions $\vec{z}(\rho,\varphi) = (z_\rho(\rho,\varphi), z_\varphi(\rho,\varphi))$, equal the zero vector on boundary S, continuous along with $\frac{\partial \vec{z}}{\partial \rho}(\rho,\varphi), \frac{\partial \vec{z}}{\partial \varphi}(\rho,\varphi)$ on Ω, in $\bar{\Omega}$, with bound $\|\vec{z}\|_{\mathring{H}_{1+\delta}(\bar{\Omega})}$. then $\|\vec{z}\|_{\mathring{H}_{1+\delta}(\bar{\Omega})} = |\vec{z}|_\delta + \left|\frac{\partial \vec{z}}{\partial \rho}\right|_\delta + \left|\frac{\partial \vec{z}}{\partial \varphi}\right|_\delta$,

where

$$|\vec{u}|_\delta = |\vec{u}|_0 + \langle \vec{u} \rangle_\delta, \quad |\vec{u}|_0 = \max_{(\rho,\varphi)\in\bar{\Omega}} |\vec{u}(\rho,\varphi)|,$$

$$\langle \vec{u} \rangle_\delta = \sup_{(\rho,\varphi),(\rho^\circ,\varphi^\circ)\in\bar{\Omega}} \frac{\left|\,|\vec{u}(\rho,\varphi)| - |\vec{u}(\rho^\circ,\varphi^\circ)|\,\right|}{\left(\sqrt{(\rho-\rho^\circ)^2 + (\varphi-\varphi^\circ)^2}\right)^\delta}.$$

We say that Vector-function $\vec{u}(\rho,\varphi) = (u_\rho(\rho,\varphi), u_\varphi(\rho,\varphi))$ satisfies in $\bar{\Omega}$ the Holder condition with the exponent $\delta, 0 < \delta \leqslant 1$ and Holder constant $\langle \vec{u} \rangle_\delta$, if $\langle \vec{u} \rangle_\delta < \infty$.

$H_\delta(\bar{\Omega})$ is a space consisting of all Vector-functions $\vec{u}(\rho,\varphi)$ continuous in $\bar{\Omega}$, with finite bound $\|\vec{u}\|_{H_\delta(\bar{\Omega})} = |\vec{u}|_0 + \langle \vec{u} \rangle_\delta$. $L_\kappa(\Omega), \kappa \geq 1$, -- is the Banach phase of Vector-functions with $\vec{f}(\rho,\varphi) = (f_\rho(\rho,\varphi), f_\varphi(\rho,\varphi))$ with $\|\vec{f}\|_{L_\kappa(\bar{\Omega})}$

$$= \max\left\{ \|f_\rho\|_{L_\kappa(\Omega)}, \|f_\varphi\|_{L_\kappa(\Omega)} \right\}, \quad \|v\|_{L_\kappa(\Omega)} = \left(\int_0^{2\pi} \int_{R_1}^{R_2} |v(\rho,\varphi)|^\kappa \rho \, d\rho \, d\varphi \right)^{\frac{1}{\kappa}}.$$

We shall denote by $\tilde{L}_\kappa(\Omega), \kappa \geq 1$, the Banach space of Vector-functions /14
$\vec{\varphi}(x_1,x_2) = (\varphi_{x_1}(x_1,x_2), \varphi_{x_2}(x_1,x_2))$ with bound

$$\|\vec{\varphi}\|_{\tilde{L}_\kappa(\Omega)} = \max\left\{ \|\varphi_{x_1}\|_{\tilde{L}_\kappa(\Omega)}, \|\varphi_{x_2}\|_{\tilde{L}_\kappa(\Omega)} \right\}, \quad \|v\|_{\tilde{L}_\kappa(\Omega)} = \left(\iint_\Omega |v(x_1,x_2)|^\kappa dx_1 dx_2 \right)^{\frac{1}{\kappa}}.$$

We shall write problem (12)-(15) in the Cartesian coordinate system.

$$\nu \Delta_{x_1,x_2} \vec{u}(x_1,x_2) = \text{deg}'_{x_1,x_2} \tilde{p}(x_1,x_2) + \vec{\varphi}(x_1,x_2), \quad (x_1,x_2) \in \Omega,$$

$$\operatorname{div}_{x_1,x_2} \vec{u} = 0, \qquad (x_1,x_2) \in \Omega,$$

$$\vec{u}\Big|_{\substack{x_1^2+x_2^2=R_2^2 \\ x_1^2+x_2^2=R_1^2}} = (0,0),$$

where $\vec{u}(x_1,x_2) = (u_{x_1}(x_1,x_2), u_{x_2}(x_1,x_2))$,

$$u_{x_1}(x_1,x_2) = \left[v_\rho(\rho,\varphi)\cos\varphi - v_\varphi(\rho,\varphi)\sin\varphi \right]_{\substack{\rho=\sqrt{x_1^2+x_2^2} \\ \varphi = \text{arc tan} \frac{x_2}{x_1}}},$$

$$u_{x_2}(x_1,x_2) = \left[v_\rho(\rho,\varphi)\sin\varphi + v_\varphi(\rho,\varphi)\cos\varphi \right]_{\substack{\rho=\sqrt{x_1^2+x_2^2} \\ \varphi = \text{arc tan} \frac{x_2}{x_1}}},$$

$$\tilde{p}(x_1,x_2)=[p(\rho,\vartheta)]_{\rho=\sqrt{x_1^2+x_2^2},\ \vartheta=\arctan\frac{x_2}{x_1}},$$

$$\vec{\varphi}(x_1,x_2)=(\varphi_{x_1}(x_1,x_2),\ \varphi_{x_2}(x_1,x_2)),$$

$$\varphi_{x_1}(x_1,x_2)=[f_\rho(\rho,\vartheta)\cos\vartheta-f_\vartheta(\rho,\vartheta)\sin\vartheta]_{\rho=\sqrt{x_1^2+x_2^2},\ \vartheta=\arctan\frac{x_2}{x_1}},$$

$$\varphi_{x_2}(x_1,x_2)=[f_\rho(\rho,\vartheta)\sin\vartheta+f_\vartheta(\rho,\vartheta)\cos\vartheta]_{\rho=\sqrt{x_1^2+x_2^2},\ \vartheta=\arctan\frac{x_2}{x_1}}$$

This boundry problem if $\vec{\varphi}(x_1,x_2)\in\tilde{L}_k(\Omega),\ k>1,$ has a unique generalized /15 solution $\vec{u}(x_1,x_2),\ \deg_{x_1,x_2}\tilde{p}(x_1,x_2)$ and the estimate $\|\operatorname{grad}_{x_1,x_2}\tilde{p}\|_{\tilde{L}_k(\Omega)}\leq q_{14}\|\vec{\varphi}\|_{\tilde{L}_k(\Omega)},$ is satisfied, where q_{14} is some number depending on k [5].

In view of the assumption relative to the Vector-function $\vec{f}(\rho,\vartheta)$, Vector-function $\vec{\varphi}(x_1,x_2)\in\tilde{L}_k(\Omega)$ for any k.

Therefore problem (12)-(15) has a unique solution $\vec{v}(\rho,\vartheta),\ \deg_{\rho,\vartheta}p(\rho,\vartheta)$ and the estimate

$$\|\operatorname{grad}_{\rho,\vartheta}p\|_{L_k(\Omega)}\leq q_{15}\|\vec{f}\|_{L_k(\Omega)}, \qquad (20)$$

is satisfied, where q_{15} is some number depending on k.

Consequently $\deg_{\rho,\vartheta}p(\rho,\vartheta)=M\vec{f}(\rho,\vartheta)$, is satisfiable, where M is a linear bounded opeator acting from space $L_k(\Omega)$ in the same space with bound $\|M\|\leq q_{15}$.

Problem (16)-(19) can be represented in the following form:

$$\begin{aligned}
&\nu\Delta_{\rho,\vartheta}\vec{v}^{(n)}-\deg_{\rho,\vartheta}p^{(n)}=Q_n\vec{f}, && (\rho,\vartheta)\in\Omega,\\
&\operatorname{div}_{\rho,\vartheta}\vec{v}^{(n)}=0, && (\rho,\vartheta)\in\Omega,\\
&\vec{v}^{(n)}\Big|_{\substack{\rho=R_1\\ \rho=R_2}}=(0,0),
\end{aligned}$$

where Q_n is the projector that placeseach coordinate of Vector-function $\vec{f}(\rho,\vartheta)$ into correspondence with its trigonometric polynomial in terms of variable of order n nodes $\vartheta_m^{(n)},\ m=0,1,\dots,2n.$

We will note that $\varphi_n \vec{f}(\rho,\psi) \in H_\delta(\bar{\Omega})$ with any n in view of the assumption /16
relative to the Vector function $\vec{f}(\rho,\psi)$. Therefore, using the same line of reasoning as above, we establish that problem (16)-(19) has a unique solution $\vec{v}^{(n)}(\rho,\psi)$, $\deg_{\rho,\psi} p(\rho,\psi)$ and the estimate

$$\begin{gathered}\|\deg_{\rho,\psi} p^{(n)}\|_{L_\kappa(\Omega)} \le q_{15}\|\varphi_n \vec{f}\|_{L_\kappa(\Omega)}, \\ \|\deg_{\rho,\psi} p^{(n)} - \deg_{\rho,\psi} p\|_{L_\kappa(\Omega)} \le q_{15}\|\vec{f} - \varphi_n \vec{f}\|_{L_\kappa(\Omega)}.\end{gathered} \tag{21}$$

is valid.

We introduce the following definition:

$$\begin{aligned}&\vec{z}(\rho,\psi) \equiv \nu\Delta_{\rho,\psi}\vec{v}, \qquad (\rho,\psi)\in\bar{\Omega}, \\ &\vec{z}^{(n)}(\rho,\psi) \equiv \nu\Delta_{\rho,\psi}\vec{v}^{(n)}, \qquad (\rho,\psi)\in\bar{\Omega}.\end{aligned} \tag{22}$$

The boundry problem is

$$\begin{aligned}&\nu\Delta_{\rho,\psi}\vec{v} = \vec{z}(\rho,\psi), \qquad (\rho,\psi)\in\Omega, \\ &\vec{v}\big|_{\substack{\rho=R_1\\ \rho=R_2}} = (0,0),\end{aligned}$$

where $\vec{z}(\rho,\psi)\in L_\kappa(\Omega)$ has a unique generalized solution $\vec{v}(\rho,\psi)$ with bound $\|\vec{v}\|_{\mathring{H}_{1+\delta}(\bar{\Omega})}$, and, as in Section 1, the estimate

$$\|\vec{v}\|_{\mathring{H}_{1+\delta}(\bar{\Omega})} \le q_{16}\|\vec{z}\|_{L_\kappa(\Omega)}, \tag{23}$$

is valid, where q_{16} is some number depending on k, δ.

Considering relations (12)-(15), (16)-(19), (22) and equation $\deg_{\rho,\psi} p = M\vec{f}$, we have the following expressions:

$$\begin{aligned}&\vec{z}(\rho,\psi) = M\vec{f}(\rho,\psi) + \vec{f}(\rho,\psi), \qquad (\rho,\psi)\in\bar{\Omega}, \\ &\vec{z}^{(n)}(\rho,\psi) = M\varphi_n\vec{f}(\rho,\psi) + \varphi_n\vec{f}(\rho,\psi), \qquad (\rho,\psi)\in\bar{\Omega}.\end{aligned}$$

We shall estimate $\|\vec{z} - \vec{z}^{(n)}\|_{L_\kappa(\Omega)}$. In view of inequality

$$\|\vec{z} - \vec{z}^{(n)}\|_{L_\kappa(\Omega)} \le (q_{15}+1)\|\vec{f} - \varphi_n\vec{f}\|_{L_\kappa(\Omega)}. \tag{24}$$

Since $\vec{f}(\rho,\varphi) \in H^{\circ}_{\delta}(\bar{\Omega})$ then according to the Berstein theorem [4]

$$\|\vec{f} - \varphi_n \vec{f}\|_{C(\bar{\Omega})} \le E_n(\vec{f}(\rho,\varphi))\,(q_{10} + q_{11}\ln n), \tag{25}$$ /17

where $E_n(\vec{f}(\rho,\varphi)) = \max\{E_n(f_\rho(\rho,\varphi)), E_n(f_\varphi(\rho,\varphi))\}$. Expressions $E_n(f_\rho(\rho,\varphi)), E_n(f_\varphi(\rho,\varphi))$ are determined in this manner in Section1. In view of note 1 $E_n(\vec{f}(\rho,\varphi)) \le q_{17}\left(\frac{2}{n}\right)^{\delta}$, where q_{17} is a constant.

The inequality

$$\|\vec{f} - \varphi_n \vec{f}\|_{L_{\varkappa}(\Omega)} \le q_{18}\,\|\vec{f} - \varphi_n \vec{f}\|_{C(\bar{\Omega})}, \tag{26}$$

is valid, where q_{18} is an absolute constant.

From inequalities (23), (25), (26) we derive

$$\|\vec{v} - \vec{v}^{(n)}\|_{\mathring{H}_{1+\delta}(\bar{\Omega})} \le q_{16}\,q_{18}\,E_n(\vec{f}(\rho,\varphi))\,(q_{10} + q_{11}\ln n).$$

Thus the following theorem is valid:

Theorem 2. If vector function $\vec{f}(\rho,\varphi) \in H_{\delta}(\bar{\Omega})$ and range $\bar{\Omega} = ((\rho,\varphi):\ 0 \le R_1 \le \rho \le R_2,\ 0 \le \varphi < 2\pi)$, then the approximate solutions $\vec{v}^{(n)}(\rho,\varphi)$, deg $_{\rho,\varphi}\, p^{(n)}(\rho,\varphi)$ bound by the collocation method, converge on the exact solution $\vec{v}(\rho,\varphi)$, deg $_{\rho,\varphi}\, p(\rho,\varphi)$ with the following estimated rate of convergence:

$$\|\vec{v} - \vec{v}^{(n)}\|_{\mathring{H}_{1+\delta}(\bar{\Omega})} \le q_{16}\,q_{17}\,q_{18}\,(q_{15}+1)\,(q_{10} + q_{11}\ln n)\left(\frac{2}{n}\right)^{\delta},$$

$$\|\nu\Delta_{\rho,\varphi}\vec{v} - \nu\Delta_{\rho,\varphi}\vec{v}^{(n)}\|_{L_{\varkappa}(\Omega)} \le q_{17}\,q_{18}\,(q_{15}+1)(q_{10} + q_{11}\ln n)\left(\frac{2}{n}\right)^{\delta},$$

$$\|\text{deg}_{\rho,\varphi}\, p - \text{grad}_{\rho,\varphi}\, p^{(n)}\|_{L_{\varkappa}(\Omega)} \le q_{17}\,q_{18}\,(q_{15}+1)(q_{10} + q_{11}\ln n)\left(\frac{2}{n}\right)^{\delta}.$$

Note 4. The results of this section pertain to the case

$$\nu\Delta_{\rho,\varphi}\vec{v} - \text{deg}_{\rho,\varphi}\, p = \vec{f}(\rho,\varphi), \quad 0 \le R_1 < \rho < R_2,\ 0 \le \varphi < 2\pi,$$
$$\text{div}_{\rho,\varphi}\,\vec{v} = 0, \qquad 0 \le R_1 < \rho < R_2,\ 0 \le \varphi < 2\pi,$$
$$\vec{v}\Big|_{\substack{\rho = R_2\\ \rho = R_1}} = (0,0),$$
$$\int_0^{2\pi} p(R_2,\varphi)\,d\varphi = 0,$$

where $\vec{r}(\rho,\varphi)=(r_\rho(\rho,\varphi), r_\varphi(\rho,\varphi), r_z(\rho,\varphi)), \vec{f}(\rho,\varphi)=(f_\rho(\rho,\varphi), f_\varphi(\rho,\varphi), f_z(\rho,\varphi))$.

Here the vector function $\vec{f}(\rho,\varphi)$ has components that satisfy the Holder condition in terms of (ρ,ϕ) in the range $\overline{\Omega} = ((\rho,\varphi):\ 0 \leq R_1 \leq \rho \leq R_2,\ 0 \leq \varphi < 2\pi)$.

3. The collocation method for quasilinear second-order parabolic equations.

We shall consider the first homogeneous boundary problem for a parabolic equation in the range $Q_T = (0,\pi) \times (0,T]$

$$\frac{\partial u}{\partial t} - Lu = f\left(x,t,u,\frac{\partial u}{\partial x}\right), \quad x \in (0,\pi),\ t \in (0,T], \qquad (27)$$

$$\text{где} \quad Lu = a_1(t)\frac{\partial^2 u}{\partial x^2} + a_2(t)\,u,$$

where

$$u(x,0) = 0, \qquad x \in (0,\pi), \qquad (28)$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \in [0,T]. \qquad (29)$$

We shall assume: 1) In the closed range $\overline{Q}_T$ $a_1(t) \geq q_{19} > 0$ is satisfied where q_{19} is a constant;

2) The coefficients of operator L are Holder-continuous with exponent δ, $\delta \in (0.1)$, in $\overline{Q}_T$:

3) The function $a_1(t)$ is Lipshits-continuous uniformly on (0,T).

We will assume that there exist the solution $u^*(x,t)$ of problem (1), (2), /19
(3), twice continuously diffentiable in terms of x and continuously differentiable in terms of t in $\overline{Q}_T$.

The following limitations are placed on the function $f\left(x,t,u,\frac{\partial u}{\partial x}\right)$

4) The function $f\left(x,t,u,\frac{\partial u}{\partial x}\right)$ is Holder-continuous with exponent δ, $0<\delta\leq 1$, relative to (x,t) uniformly in terms of $\left(u,\frac{\partial u}{\partial x}\right)$ in range $G = \left(\left(x,t,u,\frac{\partial u}{\partial x}\right):\ (x,t) \in \overline{Q}_T,\ |u - u^*(x,t)| \leq q_{20},\ \left|\frac{\partial u}{\partial x} - \frac{\partial u^*}{\partial x}(x,t)\right| \leq q_{20}\right)$, now where q_{20} is some fixed number:

5) The functions $f'_{\frac{\partial u}{\partial x}}\left(x,t,u,\frac{\partial u}{\partial x}\right)$, $f'_u\left(x,t,u,\frac{\partial u}{\partial x}\right)$ are defined and continuous in region G;

6) $f\left(0,t,0,\frac{\partial u}{\partial x}\right) = f\left(\pi,t,0,\frac{\partial u}{\partial x}\right) = 0$ при $t \in [0,T]$, $\left|\frac{\partial u}{\partial x} - \frac{\partial u^*}{\partial x}(x,t)\right| \leq q_{20}$.

We will note that conditions 1), 2), 3), and 6) insure a solution $u^*(x,t)$ with the required differential properties and sufficiently small T if the function $f(x,t,\frac{\partial u}{\partial x},u)$ is Holder-continuous in bounded subsets of the set

$$\{(x,t,u,\frac{\partial u}{\partial x}):\ (x,t)\in \bar{Q}_T,\ -\infty<u<+\infty,\ -\infty<\frac{\partial u}{\partial x}<+\infty\}$$

[6, p. 256]

If, however, we add to the conditions some requirements for the increase of the function $f(x,t,u,\frac{\partial u}{\partial x})$ in terms of the variables $u, \frac{\partial u}{\partial x}$ exists for any T.

We shall proceed to the approximate solution by the collocation method.

We shall seek the approximate solution in the form

$$u_n(x,t) = \sum_{k=1}^{n} c_{kn}(t) \sin kx.$$

The unknown functions $c_{kn}(t), k=1,\ldots,n$, according to the collocation method, are determined from the condition that equation (27) be satisfied /20
in a given system of points $x_{kn}\in[0,\pi], k=1,\ldots,n$, i.e., with a system of n ordinary first order differential equations

$$\left[\left(\frac{\partial}{\partial t}-L\right)u_n - f\right]_{x=x_{kn}} = 0, \quad k=1,\ldots,n, \tag{30}$$

for $t\in(0,T]$, with initial conditions

$$c_{kn}(0)=0, \quad k=1,\ldots,n. \tag{31}$$

Note that for $u_n(x,t)$ the initial condition (28) and boundary conditions (29) will be satisfied.

As the points of interpolation x_{kn} we take equidistant points, i.e.,

$$x_{kn} = \frac{2k\pi}{2n+1}, \quad k=1,\ldots,n.$$

The question of the solubility of system (30) and (31) and of the conversions of approximate solutions $u_n(x,t)$ to the exact solution $u^*(x,t)$ is answered with the aid of the theory of projection methods [1].

Let

$$z^*(x,t) \equiv \frac{\partial u^*(x,t)}{\partial t} - Lu^*(x,t), \quad (x,t)\in\bar{Q}_T, \tag{32}$$

$$z_n(x,t) \equiv \frac{\partial u_n(x,t)}{\partial t} - Lu_n(x,t), \quad (x,t)\in\bar{Q}_T. \tag{33}$$

In view of assumptions relative to solution $u^*(x,t)$ the function $Z^*(x,t)$ is a continuous function in $\overline{Q}_T$.

It is easy to see that the functions $Z_n(x,t)$ are equal to zero when $t = 0$, $x = 0$, $x = \pi$.

Let us introduce the following Banach spaces. The space $\tilde{C}(\overline{Q}_T)$ is the space of functions $Z(x,t)$, that are continuous in Q_T and equal to zero when $x = 0$, $x = \pi$, $t = 0$ with bound $\|z\|_{\tilde{C}(\overline{Q}_T)} = \max_{(x,t)\in\overline{Q}_T} |z(x,t)|$.

The space $\overset{\circ}{H}_{1+\delta}(\overline{\overline{Q}}_T)$ - is the space of functions $u(x,t)$, equal to zero /21
when $x=0, x=\pi, t\in[0,T]; \; x\in(0,\pi), t=0,$ continuous with $\frac{\partial u(x,t)}{\partial x}$, with bound

$$\|u\|_{\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)} = |u|_\delta + \left|\frac{\partial u}{\partial x}\right|_\delta,$$

where

$$|v|_\delta = |v|_0 + \langle v\rangle_\delta, \quad |v|_0 = \max_{(x,t)\in\overline{Q}_T} |v(x,t)|,$$

$$\langle v\rangle_\delta = \sup_{(x,t),(x^0,t^0)\in\overline{Q}_T} \frac{|v(x,t)-v(x^0,t^0)|}{(|x-x^0|^2+|t-t^0|)^{\frac{\delta}{2}}}.$$

We shall examine in space $\tilde{C}(\overline{Q}_T)$ in linear variety $\tilde{C}(\overline{Q}_T)$ of functions $Z(x,t)$, satisfying the Holder condition with exponent condition δ_0, $\delta_0 < \delta$, in set Q_T. This set $\tilde{C}(\overline{Q}_T)$ is everywhere continuous in space $\tilde{C}(\overline{Q}_T)$.

The boundary problem is

$$\begin{aligned} &\frac{\partial u}{\partial t} - Lu = z(x,t), && (x,t)\in Q_T,\\ &u(x,0)=0, && x\in(0,\pi),\\ &u(0,t)=u(\pi,t)=0, && t\in[0,T], \end{aligned}$$

if the function $z(x,t)\in\widetilde{\widetilde{C}}(\overline{Q}_T)$ with conditions 1), 2), 3), satisfied, has a unique classical solution $u(x,t)$ then

$$\|u\|_{\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)} \le q_{21}\|z\|_{\tilde{C}(\overline{Q}_T)}, \tag{34}$$

where q_{21} is a constant [6].

In other words, we have determined a linear bounded operator D, acting from $\tilde{C}(\overline{Q}_T)$ in $\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)$ with bound $\|D\| \le q_{21}$. Since $\tilde{C}(\overline{Q}_T)$ is everywhere continuous

$\tilde{C}(\bar{Q}_T)$, operator D can be expanded over an entire space $\tilde{C}(\bar{Q}_T)$ with the same bound.

Thus, if the functions $z_n(x,t) \in \tilde{C}(\bar{Q}_T), n=1,\ldots$ for the function $u^*(x,t)-u_n(x,t)$, $z^*(x,t)-z_n(x,t)$ in view of (34), the inequalities /22

$$\|u_n - u^*\|_{\mathring{H}_{1+\delta}(\bar{Q}_T)} \le q_{21}\|z_n - z^*\|_{\tilde{C}(\bar{Q}_T)}, \quad n = 1, 2, \ldots \tag{35}$$

will be satisfied.

We shall use $\mathring{H}_{\delta_0}(\bar{Q}_T)$ to denote a Banach space of function u(x,t) equal to zero when $x=0, x=\pi, t\in[0,T]$ and satisfying the Holder condition in $\bar{Q}_T$ with δ_0, $0<\delta_0\le 1$. The bound $\|u\|_{\mathring{H}_{\delta_0}(\bar{Q}_T)}$ is defined by the relationship $\|u\|_{\mathring{H}_{\delta_0}(\bar{Q}_T)} = |u|_0 + \langle u\rangle_{\delta_0}$. Expressions $\langle\cdot\rangle_n$, $|\cdot|_n$ were determined above.

Since operator D is bounded we can take in space $\tilde{C}(\bar{Q}_T)$ the sphere $\|z-z^*\|_{\tilde{C}(\bar{Q}_T)} \le \delta_3$ of radius δ_3 so small that the functions u(x,t) = D Z(x,t), $\|z-z^*\|_{\tilde{C}(\bar{Q}_T)} \le \delta_3$ will satisfy the inequalities:

$$|u-u^*(x,t)| \le q_{20}, \quad \left|\frac{\partial u}{\partial x} - \frac{\partial u^*(x,t)}{\partial x}\right| \le q_{20}, \quad (x,t)\in \bar{Q}_T.$$

We proceed from problem (27)-(29) to that of finding the function Z*(x,t), belonging to space $\tilde{C}(\bar{Q}_T)$, satisfying the condition

$$z(x,t) = K f\left(x,t,Dz(x,t),\frac{\partial}{\partial x}Dz(x,t)\right) = KNz(x,t). \tag{36}$$

Here K is the linear bounded operator of enclosure of $\mathring{H}_{\delta_0}(\bar{Q}_T)$ in $\tilde{C}(\bar{Q}_T)$, $N = f(x,t,D\ldots,\frac{\partial}{\partial x}D\ldots)$ is the operator acting from set $Z = (z: \|z-z^*\|_{\tilde{C}(\bar{Q}_T)} \le \delta_3) \subset \tilde{C}(\bar{Q}_T)$ in space $\mathring{H}_{\delta_0}(\bar{Q}_T)$, where $\delta_0 < \delta$.

The operator changes functions from the set $Z \subset \tilde{C}(\bar{Q}_T)$ into function belonging to $\mathring{H}_{\delta}(\bar{Q}_T)$, since operator D acts from $\tilde{C}(\bar{Q}_T)$ in $\mathring{H}_{1+\delta}(\bar{Q}_T)$ and the function $f(x,t,u,\frac{\partial u}{\partial x})$ satisfies condition 4). Space $\mathring{H}_{\delta}(\bar{Q}_T)$ fits compactly /23
in space $\mathring{H}_{\delta_0}(\bar{Q}_T)$, consequently operator N is perfectly continuous.

We will note that $z^* \in \tilde{C}(\bar{Q}_T)$ follows from $z^* \in \tilde{C}(\bar{Q}_T),\ z^* = KNz^*$ since KN changes from set $Z \subset\subset \tilde{C}(\bar{Q}_T)$ to $\mathring{H}_{\delta_0}(\bar{Q}_T) \subseteq \tilde{C}(\bar{Q}_T)$.

Consequently, if u*(x,t) is a solution to (27)-(29), twice continuously differentiable in terms of x, continuously differentiable in terms of t in $\bar{Q}_T$, then $z^*(x,t) \equiv \frac{\partial u^*(x,t)}{\partial t} - \mathcal{L}u^*(x,t)$ is the solution of problem (36), $z^*(x,t) \in \tilde{C}(\bar{Q}_T)$ and conversely if Z*(x,t) is the solution of equation (36), the function u*(x,t) found from boundary problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \mathcal{L}u = z^*(x,t), && (x,t) \in \bar{Q}_T, \\ &u(x,0) = 0, && x \in (0,\pi), \\ &u(0,t) = u(\pi,t) = 0, && t \in [0,T], \end{aligned}$$

will be twice continuously differentiable in terms of x, continuously differentiable in terms of t in $\bar{Q}_T$ and will be the solution of problem (27)-(29).

From problem (30), (31) of determining the approximate solution of $u_n(x,t)$ we will proceed to the problem of finding the function that satisfies the operator equation

$$K_n\ (z_n - f(x, t, \mathcal{D}z_n, \tfrac{\partial}{\partial x}\mathcal{D}z_n)) = 0, \tag{37}$$

where K_n is the porjector that places each continuous function $\Psi(x,t)$ to correspond with its interpolation trigonometric polynomial of order of n with node $x_{kn},\ k = 1,\dots,n$. However $z_n(x,t) = \frac{\partial u_n(x,t)}{\partial t} - \mathcal{L}u_n(x,t)$ - is a polynomial of order not greater then n in terms of argument x, and this means $KZ_n = Z_n$, and from equation (37) we proceed to the equation

$$z_n = K_n f(x, t, \mathcal{D}z_n, \tfrac{\partial}{\partial x}\mathcal{D}z_n) = K_n N z_n\,. \tag{38}$$

Linear bounded operator K_n acts from $\mathring{H}_{\delta_0}(\bar{Q}_T)$ to $\tilde{C}(\bar{Q}_T)$. Since operator K_nN converts $Z \subset \tilde{C}(\bar{Q}_T)$ to $\mathring{H}_{\delta}(\bar{Q}_T) \subseteq \tilde{C}(\bar{Q}_T)$. $z_n^* \in \tilde{C}(\bar{Q}_T)$. follows from $z_n^* \in \tilde{C}(\bar{Q}_T),\ z_n^* = K_nNz_n^*$.

From this we derive the relationships of problems (30), (31) and (38), analogous to that of problem (27)-(29) and (36).

In view of interpolation theorem [4] for any $z \in \mathring{H}_{\delta_0}(\bar{Q}_T)$ .we have $\|K_n z - Kz\|_{\tilde{C}(\bar{Q}_T)} \to 0$ for $n \to +\infty$.

In view of conditions 5) operator KN is continuously Freshe-differentiable at the point $Z^*(x,t)$ in space $\tilde{C}(\overline{Q}_T)$.

We will prove that the homogeneous equation $h=KN'(z^*)h$ has only a trivial solution. If $h \in \tilde{C}(\overline{Q}_T)$, then equation $h=KN'(z^*)h$ is equivalent to the boundary problem

$$\frac{\partial u}{\partial t} - Lu = f'_u\left(x,t,u^*(x,t),\frac{\partial u^*}{\partial x}(x,t)\right)u + f'_{\frac{\partial u}{\partial x}}\left(x,t,u^*(x,t),\frac{\partial u^*}{\partial x}(x,t)\right)\frac{\partial u}{\partial x}, \qquad (x,t)\in Q_T;$$

$$u(x,0)=0, \quad x\in(0,\pi); \qquad u(0,t)=u(\pi,t)=0, \quad t\in[0,T].$$

This homogeneous problem has only a zero solution. If $h_1 \in \tilde{C}(\overline{Q}_T)$, $h_1 = KN'(z^*)h_1$. is satisfied it is essential that $h_1 \in \tilde{C}(\overline{Q}_T)$. Hence, $h_1 = 0$.

Uner the conditions stipulated in this section the theorem concerning the convergence of approximate solutions to the exact solutions is applicable [1, pp. 293-294].

In view of this theorem we also find numbers n_1, σ_1, such that when $n \geqslant n_1$ there is a unique solution $Z_n(x,t)$ of equation (38) in the sphere $\|z-z^*\|_{\tilde{C}(\overline{Q}_T)} \leq \sigma_1$ all such approximate s olutions $Z_n(s,t)$ converge on $Z^*(x,t)$ in the bound of space $\tilde{C}(\overline{Q}_T)$ and the estimate of convergence

$$q_{22}\|z^*-K_n z^*\|_{\tilde{C}(\overline{Q}_T)} \leq \|z_n-z^*\|_{\tilde{C}(\overline{Q}_T)} \leq q_{23}\|z^*-K_n z^*\|_{\tilde{C}(\overline{Q}_T)} \qquad (25)$$

is valid where q_{22}, q_{23} are constant.

From interpolation theorem [4] we derive $\|z^*-K_n z^*\|_{\tilde{C}(\overline{Q}_T)} \leq E_n(z^*(x,t))(q_{24}+q_{25}\ln n)$, where $E_n(z^*(x,t)) = \sup_{t\in[0,T]} E_n^x(z^*(x,t))$, $E_n^x(z^*(x,t))$ is the best uniform approximation of the function $Z^*(x,t)$ by the trigonometric polynomials of order not exceeding n in terms of variable x for fixed t, $t\in[0,T]$; q_{24}, q_{25} - are absolute constants.

To each solution $Z_n(x,t)$ of problem (38) there corresponds a solution $U_n(x,t)$ of problem (30), (31) in view of estimate (35)

$$\|u_n-u^*\|_{A_{1+\delta}(\overline{Q}_T)} \leq q_{21}\|z_n-z^*\|_{\tilde{C}(\overline{Q}_T)} \leq q_{21}q_{23}\|z^*-K_n z^*\|_{\tilde{C}(\overline{Q}_T)}.$$

Consequently the following theorem is valid:

Theorem 3. When all the functions of this section are satisfied numbers n_1, σ_4, are found such that when $n \geqslant n_4$ there exist unique approximate solutions $U_n(x,t)$ in sphere $\| u - u^* \|_{\overset{\circ}{H}_{1+\delta}(\bar{Q}_T)} \leq q_{21}\sigma_4$ and the estimates

$$\| u_n - u^* \|_{\overset{\circ}{H}_{1+\delta}(\bar{Q}_T)} \leq q_{21} q_{23} E_n\left(\left(\frac{\partial}{\partial t} - L\right) u^*(x,t)\right)(q_{24} + q_{25} \ln n),$$

$$q_{22} \left\| \left(\frac{\partial}{\partial t} - L\right) u^* - K_n \left(\frac{\partial}{\partial t} - L\right) u^* \right\|_{\tilde{C}(\bar{Q}_T)} \leq \left\| \left(\frac{\partial}{\partial t} - L\right) u^* - \left(\frac{\partial}{\partial t} - L\right) u_n \right\|_{\tilde{C}(\bar{Q}_T)} \leq$$

$$\leq q_{23} \left\| \left(\frac{\partial}{\partial t} - L\right) u^* - K_n \left(\frac{\partial}{\partial t} - L\right) u^* \right\|_{\tilde{C}(\bar{Q}_T)} \leq$$

$$\leq q_{23} E_n\left(\left(\frac{\partial}{\partial t} - L\right) u^*(x,t)\right)(q_{24} + q_{25} \ln n) .$$

are satisfied.

Note 5. If the function $\frac{\partial u^*}{\partial t}(x,t) - Lu^*(x,t)$ has continuous derivative /26
$\frac{\partial^s}{\partial x^s}\left[\left(\frac{\partial}{\partial t} - L\right) u^*(x,t)\right]$, $s = 1, 2, \ldots,$ when satisfying in terms of x the Holder condition with index α, $0 < \alpha \leq 1$, uniform with respect to t, then according to the Jackson theorem [4] $E_n\left(\frac{\partial u^*}{\partial t}(x,t) - Lu^*(x,t)\right) \leq q_{13} q_{26} \frac{(s+1)^{s+1}}{(s+1)!} \frac{2^\alpha}{(n-s)^\alpha n^s}$, where q_{26} is the Holder constant of the function $\frac{\partial^s}{\partial x^s}\left[\left(\frac{\partial}{\partial t} - L\right) u^*(x,t)\right]$.

Note 6. If $x \in [a, b]$, the results of this section remain unchanged after linear substitution of variables $y = \frac{\pi}{b-a}(x - a)$.

4. Collocation method for nonstationary Navier-Stokes equation system for dynamic viscosity of incompressible fluid.

We shall examine the following boundary problem in the range $Q_T = \{(x,t) : a < x < b,\ 0 < t \leq T\}$

$$\frac{\partial \vec{v}}{\partial t}(x,t) - \nu \frac{\partial^2 \vec{v}(x,t)}{\partial x^2} = \text{deg}\ p(x,t) + \vec{f}(x,t), \quad (x,t) \in Q_T, \tag{39}$$

where

$$\vec{v}(x,t) = (v_x(x,t), v_y(x,t), v_z(x,t)), \quad \text{grad}\ p(x,t) =$$

$$= \left(\frac{\partial p}{\partial x}(x,t), 0, 0\right), \quad \vec{f}(x,t) = (f_x(x,t), f_y(x,t), f_z(x,t)), \tag{40}$$

$$\frac{\partial v_x}{\partial x}(x,t) = 0, \qquad (x,t) \in Q_T, \tag{41}$$

$$\vec{v}(a,t) = \vec{v}(b,t) = (0,0,0), \quad t \in [0,T], \tag{42}$$

$$v(x,0) = (0,0,0), \qquad x \in (a,b)$$

It follows from relation (40) and (41) that $v_x(x,t) \equiv 0$. Systems (39), (42) are broken down into independent equations for determining the functions deg $p(x,t)$, $v_y(x,t)$, $v_z(x,t)$.

We will assume that the functions $f_y(x,t)$, $f_z(x,t)$ are Holder-continuous /27
with δ, $0 < \delta \leq 1$, in closed range $\overline{Q}_T$. Furthermore, $f_y(a,t) = f_y(b,t) = f_z(a,t) = f_z(b,t) = 0$ when $t \in [0,T]$.

The approximate solution $v_y^{(n)}(x,t)$, $v_z^{(n)}(x,t)$ is sought in the form

$$v_y^{(n)}(x,t) = \sum_{k=1}^{n} \lambda_{kn}(t) \sin k\pi \frac{(x-a)}{b-a},$$
$$v_z^{(n)}(x,t) = \sum_{k=1}^{n} \tau_{kn}(t) \sin k\pi \frac{(x-a)}{b-a}$$

The unknown functions $\lambda_{kn}(t)$, $\tau_{kn}(t)$, $k = 1, \ldots, n$, are determined from the condition that equation (39) is satisfied in the given system of points $x_{kn} \in [a,b]$, i.e., from the system of 2n first-order differential equations. We will assume $x_{kn} = \frac{2k(b-a)}{2n+1} - a$, $k = 1, \ldots, n$

There exists the unique solution $v_y^*(x,t)$, $v_z^*(x,t)$ of problem (39)-(42) in view of the assumptions relative to the functions $f_y(x,t)$, $f_z(x,t)$ (ε).

Using the results of the proceding section we arrive at the following theorem:

Theorem 4. Let the functions $f_y(x,t)$, $f_z(x,t)$ satisfy the state of this condition.

Then for sufficiently large n there exist unique approximate solutions $v_y^{(n)}(x,t)$, $v_z^{(n)}(x,t)$ and $\| v_y^{(n)} - v_y^* \|_{\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)} \to 0$, $\| v_z^{(n)} - v_z^* \|_{\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)} \to 0$ for $n \to +\infty$.

The rate of convergence is characterized by the inequalities

$$\| v_y^{(n)} - v_y^* \|_{\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)} \leq q_{27} q_{28} \frac{(q_{24} + q_{25} \ln n)}{n^{\delta}},$$
$$\| v_z^{(n)} - v_z^* \|_{\overset{\circ}{H}_{1+\delta}(\overline{Q}_T)} \leq q_{29} q_{28} \frac{(q_{24} + q_{25} \ln n)}{n^{\delta}},$$

$$\left\|\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)z_y^{(n)}-\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)z_y^{*}\right\|_{\tilde{C}(\bar{Q}_T)}\leq\frac{q_{27}q_{30}(q_{24}+q_{25}\ln n)}{n^{\delta}},$$

$$\left\|\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)z_z^{(n)}-\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)z_z^{*}\right\|_{\tilde{C}(\bar{Q}_T)}\leq\frac{q_{29}q_{30}(q_{24}+q_{25}\ln n)}{n^{\delta}}.$$ /28

Here q_{27}, q_{29} are the Holder constants of the functions $f_y(x,t)$, $f_z(x,t)$ respectively, and q_{28}, q_{30} are absolute constants. The bounds $\|u\|_{\mathring{H}_{1+\delta}(\bar{Q}_T)}$, $\|u\|_{\tilde{C}(\bar{Q}_T)}$ are defined in Section 3.

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